J. Fluid Mech. (2002), *vol.* 450, *pp.* 287–296. © 2002 Cambridge University Press 287
DOI: 10.1017/S0022112001006620 Printed in the United Kingdom

A new model for shallow water in the low-Rossby-number limit

By RUPERT FORD[1]†, SIMON J. A. MALHAM[2] AND MARCEL OLIVER[3]

[1]Department of Mathematics, Imperial College, London SW7 2BZ, UK

[2]Department of Mathematics, Heriot–Watt University, Edinburgh EH14 4AS, UK
e-mail: S.J.Malham@ma.hw.ac.uk

[3]Mathematisches Institut, Universität Tübingen, 72076 Tübingen, Germany
e-mail: oliver@member.ams.org

(Received 31 July 2001 and in revised form 19 September 2001)

We revisit Salmon's 'Dirac bracket projection' approach to constructing generalized semi-geostrophic equations. One of the obstacles to the method's applicability is that it leads to a sign-indefinite energy functional in the computational domain. In some instances this can cause severe failure of the model. We demonstrate in the simple context of shallow-water semi-geostrophy that the Hamiltonian remains positive definite when the asymptotic expansion at the heart of this method is carried to the next order. The resulting new model can be interpreted in the framework of regularization by Lagrangian averaging, which is currently receiving much attention.

1. Introduction

In a series of articles, Salmon proposed new approximate models for nearly geostrophic flow in a layer of shallow water (1985, 1988), and in a layer of stratified fluid of finite depth (1996). The models were subsequently described from a more general perspective by McIntyre & Roulstone (1996, 2001). Their derivation is an example of *variational asymptotics*: all approximations are performed on the Lagrangian of the parent fluid model before Hamilton's principle is applied to yield new equations of motion. One of the chief advantages of this approach is that preservation of time and particle relabelling symmetries guarantees exact conservation of a new energy and potential vorticity in the approximate system.

Salmon's approximation consists of two steps. First, noting that the stationary leading-order geostrophic balance defines a submanifold in phase space, he constrains the full Lagrangian to this 'slow manifold'. This step, however, destroys the canonical coordinate structure of the variational formulation. The second step is therefore a transformation back to simple canonical variables. This is feasible only approximately to some order in the formal small parameter.

While built-in structure preservation is clearly an attractive feature, the conservation laws of the new system may not necessarily be physically reasonable. In particular, the energy may not be sign definite, which can destroy consistency of the evolution with the underlying model assumptions, and have a serious impact on numerical simulation, as has been observed by Shepherd & Ford (2001).

† Rupert Ford passed away on March 30, 2001, while this article was being prepared.

The purpose of this paper is to demonstrate that loss of definiteness is not necessarily associated with Salmon's method. We show that by going to the next order in the approximation of the transformation to canonical coordinates, positivity is preserved. This is consistent with the idea that the transformed system should stay asymptotically closer to the 'slow manifold' than the original system. We consider only the shallow-water situation. The stratified case is much more subtle, and will be considered in a forthcoming paper. Our basic idea, however, applies to the stratified case as well.

The paper is laid out as follows. We first recall the shallow-water model and sketch the nearly geostrophic large-scale approximation following the explanation given in Salmon (1988). We then introduce new notation which allows us to more systematically keep track of second-order quantities. The crucial point is that we treat the change of coordinates as a flow with respect to the perturbation parameter, and expand the Lagrangian as an asymptotic series in terms of the expansion of the vector field generated by this flow. This setup was previously used in, and is indeed inspired by, Marsden & Shkoller's (2000, 2001) theory of Lagrangian averaging. The computation in the main part of the paper retraces the steps of Salmon including terms of second order in the Rossby number, thereby restoring definiteness to the Hamiltonian. The paper concludes with the derivation of the associated evolution equation, and a discussion of the larger context of the result.

2. Models for nearly geostrophic shallow water

We consider an infinitely extended layer of rotating shallow water whose horizontal velocity $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and fluid depth $h = h(\boldsymbol{x}, t)$ is governed by the shallow-water equations

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\, \boldsymbol{u}^\perp + g\, \nabla h = 0, \tag{2.1a}$$

$$\partial_t h + \nabla \cdot (h \boldsymbol{u}) = 0, \tag{2.1b}$$

where $\boldsymbol{u}^\perp = (-u_2, u_1)$, f is the Coriolis parameter, and g the acceleration due to gravity. We assume that h approaches a constant, and $\boldsymbol{u}$ vanishes at infinity. For simplicity, we consider only the case of constant Coriolis parameter, though we believe that the calculation can be extended to the general case.

We are interested in nearly geostrophic flow, i.e. the case of small Rossby number

$$\varepsilon = \frac{U}{fL} \ll 1, \tag{2.2}$$

and, as in Salmon (1985), small Burger number

$$B = \frac{gH}{f^2 L^2} = \varepsilon, \tag{2.3}$$

where H is the mean layer depth, U the horizontal velocity scale, and L the horizontal geometric length scale. Then, assuming an advective time scale, the non-dimensionalized shallow-water equations read

$$\varepsilon\, (\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \boldsymbol{u}^\perp + \nabla h = 0, \tag{2.4a}$$

$$\partial_t h + \nabla \cdot (h \boldsymbol{u}) = 0. \tag{2.4b}$$

Our goal is to construct a hierarchy of Hamiltonian balanced models using ε as a formal expansion parameter. At the lowest order $\varepsilon = 0$, we have the geostrophic balance relation

$$\boldsymbol{u}_{\text{geostrophic}} = \nabla^{\perp} h. \tag{2.5}$$

Substituting (2.5) back into the continuity equation (2.4b) we see that the Eulerian dynamics is stationary to leading order. More generally, we shall obtain equations of motion of the form

$$\partial_t h + \nabla^{\perp} \cdot (\phi \, \nabla h) = 0, \tag{2.6}$$

where, up to terms of second order in ε,

$$\phi = h + \tfrac{1}{2}\, \varepsilon \, |\nabla h|^2 + \varepsilon \, h \, \Delta h + \varepsilon^2 \, h \, |\text{Hess}\, h|^2 + \varepsilon^2 \, \text{Hess} : (h^2 \, \text{Hess}\, h); \tag{2.7}$$

Hess denotes the Hessian acting on scalar functions as a matrix-valued second-order differential operator, and the colon denotes contraction over two indices; in other words, Hess : $(h^2 \, \text{Hess}\, h) = \partial_{ij}(h^2 \, \partial_{ij} h)$.

Truncated to first order, this is precisely Salmon's (1985) large-scale semi-geostrophic model. Moreover, the reduced models are Hamiltonian, as are the parent shallow-water equations, with respect to the energy

$$H = \tfrac{1}{2} \int \left[h^2 - \varepsilon \, h \, |\nabla h|^2 + \varepsilon^2 \, h^2 \, |\text{Hess}\, h|^2 \right] \, \mathrm{d}\boldsymbol{x}. \tag{2.8}$$

A simple computation shows that the negative first-order term is 'squeezed' in between the zero and second order:

$$\begin{aligned} \varepsilon \int h \, |\nabla h|^2 \, \mathrm{d}\boldsymbol{x} &= -\frac{\varepsilon}{2} \int h^2 \, \Delta h \, \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{2} \left(\int h^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\varepsilon^2 \int h^2 \, (\Delta h)^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2} \\ &\leqslant \frac{1}{4} \int h^2 \, \mathrm{d}\boldsymbol{x} + \frac{\varepsilon^2}{4} \int h^2 \, (\Delta h)^2 \, \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{4} \int h^2 \, \mathrm{d}\boldsymbol{x} + \frac{\varepsilon^2}{4} \int h^2 \, |\text{Hess}\, h|^2 \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{2.9}$$

Hence H is positive definite provided the layer depth is strictly positive. Since h is an advected scalar, this only amounts to requiring that the initial data be consistent with the underlying model hypotheses.

3. Variational approach to semi-geostrophic theory

The derivation of 2.6 is based on the observation that the full rescaled shallow-water equations can be obtained via Hamilton's principle from the Lagrangian

$$L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \int (\boldsymbol{P} + \varepsilon \, \boldsymbol{u}) \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} \, \mathrm{d}\boldsymbol{a} - \tfrac{1}{2} \int (\varepsilon \, |\boldsymbol{u}|^2 + h) \circ \boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a}, \tag{3.1}$$

where $\boldsymbol{\eta}$ is the map from Lagrangian labels $\boldsymbol{a}$ to Eulerian particle positions $\boldsymbol{x}$, $\boldsymbol{P}$ is the two-dimensional 'vector potential' of the Coriolis parameter, i.e. $\nabla^{\perp} \cdot \boldsymbol{P} = f \equiv 1$, and $\circ$ denotes composition of maps so that, in particular, $\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta}$.

By relating the Lagrangian phase-space coordinates $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$ to Hamiltonian canonical variables $\boldsymbol{p}$ and $\boldsymbol{q}$ through the Legendre transform, which in our situation takes the trivial form

$$(\boldsymbol{\eta}, \varepsilon \, \dot{\boldsymbol{\eta}}) \mapsto (\boldsymbol{q}, \boldsymbol{p}), \tag{3.2}$$

we notice that the Lagrangian is of the form

$$L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \int (\boldsymbol{P} + \boldsymbol{p}) \cdot \dot{\boldsymbol{q}} \, \mathrm{d}\boldsymbol{a} - H(\boldsymbol{p}, \boldsymbol{q}), \tag{3.3}$$

and that Hamilton's equations of motion can be written

$$\begin{pmatrix} -\boldsymbol{J} & -\boldsymbol{I} \\ \boldsymbol{I} & 0 \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} \delta H/\delta \boldsymbol{q} \\ \delta H/\delta \boldsymbol{p} \end{pmatrix} \qquad \text{where } \boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

The matrix on the left is clearly symplectic.

Now the basic idea is to view the zero-order geostrophic balance (2.5), being of the form

$$\boldsymbol{p} = G(\boldsymbol{q}), \tag{3.5}$$

as defining a 'slow manifold' in phase space. In fact, when $\varepsilon = 0$ the dynamics on this manifold is stationary. For $\varepsilon > 0$ small, we seek an approximate solution by projecting the full dynamics onto the 'slow manifold,' i.e. we add (3.5) as a constraint to the variational principle. This fits well into the Dirac theory of constraints – thus the term *Dirac bracket projection* (Salmon 1988). Here the situation is even simpler: the constraint is the graph of an operator, so we can simply insert (3.5) into the Lagrangian (3.3).

An important consequence of the general theory is that the constrained system again is Hamiltonian. It is not, however, in canonical form because $\boldsymbol{p}$ is no longer a phase-space coordinate. Moreover, a transformation to canonical coordinates, albeit known to exist, cannot easily be found. As a way out, Salmon noticed that it is possible to pose an approximate transformation to canonical form that incurs errors only of the same formal order as those introduced by the projection itself. Equivalently, we can say that changing the slow manifold by a small amount enables us to find an explicit transformation of the constrained system to canonical coordinates.

In the following we revisit the details of this construction, and demonstrate that by keeping the transformation to canonical coordinates one order more accurate than the projection, we preserve definiteness of the Hamiltonian.

4. The setup

Let $\boldsymbol{u}_\varepsilon$ denote the velocity in physical coordinates, and $\boldsymbol{u}$ the velocity in the new semi-geostrophic coordinate system. We write the old, unconstrained Lagrangian in this notation:

$$L_\varepsilon = \int (\boldsymbol{P} + \varepsilon \boldsymbol{u}_\varepsilon) \circ \boldsymbol{\eta}_\varepsilon \cdot \dot{\boldsymbol{\eta}}_\varepsilon \, \mathrm{d}\boldsymbol{a} - H_\varepsilon, \tag{4.1}$$

$$H_\varepsilon = \tfrac{1}{2} \int \left(\varepsilon \, |\boldsymbol{u}_\varepsilon|^2 + h_\varepsilon\right) \circ \boldsymbol{\eta}_\varepsilon \, \mathrm{d}\boldsymbol{a}. \tag{4.2}$$

The layer depth in physical coordinates, usually written $h_\varepsilon = \partial \boldsymbol{a}/\partial \boldsymbol{x}_\varepsilon$, is the Jacobian of the change from Eulerian to Lagrangian coordinates. In 'fixed-slot' notation which, in our experience, is less prone to error, we write

$$h_\varepsilon^{-1} \circ \boldsymbol{\eta}_\varepsilon \equiv J_\varepsilon = \det \nabla \boldsymbol{\eta}_\varepsilon. \tag{4.3}$$

The flow in each coordinate system has an associated vector field via

$$\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta}, \tag{4.4}$$

$$\dot{\boldsymbol{\eta}}_\varepsilon = \boldsymbol{u}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon. \tag{4.5}$$

The change of coordinates can be expressed as a transformation:

$$\boldsymbol{\eta}_\varepsilon = \boldsymbol{\xi}_\varepsilon \circ \boldsymbol{\eta}. \tag{4.6}$$

At this stage all objects are still flow maps, and there is no truncation to some order of ε yet. The crucial point is that we can regard $\boldsymbol{\xi}_\varepsilon$ as a flow in ε, and associate with it a vector field $\boldsymbol{v}_\varepsilon$ via

$$\boldsymbol{\xi}_\varepsilon' = \boldsymbol{v}_\varepsilon \circ \boldsymbol{\xi}_\varepsilon, \tag{4.7}$$

where the prime denotes a derivative with respect to ε. The computations which follow are more easily written in terms of the Eulerian vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$, so we need to establish a few identities between derivatives of the diffeomorphisms and the corresponding vector fields. Differentiating (4.7) with respect to t and ε, respectively, gives

$$\dot{\boldsymbol{\xi}}_\varepsilon' = \dot{\boldsymbol{v}}_\varepsilon \circ \boldsymbol{\xi}_\varepsilon + (\nabla \boldsymbol{v}_\varepsilon) \circ \boldsymbol{\xi}_\varepsilon \, \dot{\boldsymbol{\xi}}_\varepsilon, \tag{4.8}$$

$$\boldsymbol{\xi}_\varepsilon'' = \boldsymbol{v}_\varepsilon' \circ \boldsymbol{\xi}_\varepsilon + (\nabla \boldsymbol{v}_\varepsilon) \circ \boldsymbol{\xi}_\varepsilon \, \boldsymbol{\xi}_\varepsilon'. \tag{4.9}$$

Setting $\varepsilon = 0$ and using that, by definition, $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_0 = \mathrm{id}$ and therefore $\dot{\boldsymbol{\xi}} = 0$, we obtain

$$\boldsymbol{\xi}' = \boldsymbol{v}, \tag{4.10}$$

$$\dot{\boldsymbol{\xi}}' = \dot{\boldsymbol{v}}, \tag{4.11}$$

$$\boldsymbol{\xi}'' = \boldsymbol{v}' + \nabla \boldsymbol{v} \, \boldsymbol{v}. \tag{4.12}$$

(Quantities without subscript are taken to be evaluated at $\varepsilon = 0$.) Similarly, successive differentiation of (4.6) gives

$$\boldsymbol{\eta}_\varepsilon' = \boldsymbol{\xi}_\varepsilon' \circ \boldsymbol{\eta}, \tag{4.13}$$

$$\boldsymbol{\eta}_\varepsilon'' = \boldsymbol{\xi}_\varepsilon'' \circ \boldsymbol{\eta}, \tag{4.14}$$

$$\dot{\boldsymbol{\eta}}_\varepsilon' = \dot{\boldsymbol{\xi}}_\varepsilon' \circ \boldsymbol{\eta} + (\nabla \boldsymbol{\xi}_\varepsilon') \circ \boldsymbol{\eta} \, \dot{\boldsymbol{\eta}}, \tag{4.15}$$

whence, setting $\varepsilon = 0$,

$$\boldsymbol{\eta}' = \boldsymbol{v} \circ \boldsymbol{\eta}, \tag{4.16}$$

$$\boldsymbol{\eta}'' = (\boldsymbol{v}' + \nabla \boldsymbol{v} \, \boldsymbol{v}) \circ \boldsymbol{\eta}, \tag{4.17}$$

$$\dot{\boldsymbol{\eta}}' = (\dot{\boldsymbol{v}} + \nabla \boldsymbol{v} \, \boldsymbol{u}) \circ \boldsymbol{\eta}. \tag{4.18}$$

We remark that, although we work explicitly in Euclidean coordinates, these expressions could easily be written in intrinsic geometric notation.

5. Restriction to the constraint manifold

In the notation of the previous section, the constraint to the geostrophic manifold is applied in the old, ε-indexed variables:

$$\boldsymbol{u}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon = (\nabla^\perp h_\varepsilon) \circ \boldsymbol{\eta}_\varepsilon = (\nabla^\perp h + \varepsilon \, \boldsymbol{w}) \circ \boldsymbol{\eta} + O(\varepsilon^2), \tag{5.1}$$

where $\boldsymbol{w}$ is implicitly defined through the $O(\varepsilon)$ terms in the expansion of $(\nabla^\perp h_\varepsilon) \circ \boldsymbol{\eta}_\varepsilon$. As it turns out, we will never need to compute $\boldsymbol{w}$ explicitly, but we could certainly do so. There are two terms in the Lagrangian in which substitution (5.1) occurs, namely

$$\begin{aligned} \varepsilon \, \boldsymbol{u}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \cdot \dot{\boldsymbol{\eta}}_\varepsilon &= \varepsilon \, \left(\nabla^\perp h + \varepsilon \, \boldsymbol{w}\right) \cdot \left(\dot{\boldsymbol{\eta}} + \varepsilon \, \dot{\boldsymbol{\eta}}'\right) + O(\varepsilon^3) \\ &= \varepsilon \, \nabla^\perp h \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \varepsilon^2 \, \left(\boldsymbol{w} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \nabla^\perp h \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}}'\right) + O(\varepsilon^3) \end{aligned} \tag{5.2}$$

and

$$\varepsilon \, |\boldsymbol{u}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon|^2 = \varepsilon \, |\nabla h \circ \boldsymbol{\eta}|^2 + 2 \, \varepsilon^2 \, (\boldsymbol{w} \cdot \nabla^\perp h) \circ \boldsymbol{\eta} + O(\varepsilon^3). \tag{5.3}$$

6. Transformation of the Lagrangian

Consider the first term of the Lagrangian. As f is constant, second derivatives of $\boldsymbol{P}$ vanish, and a straightforward Taylor expansion of $\boldsymbol{P} \circ \boldsymbol{\eta}_\varepsilon$ about $\varepsilon = 0$ gives

$$\boldsymbol{P} \circ \boldsymbol{\eta}_\varepsilon = \boldsymbol{P} \circ \boldsymbol{\eta} + \varepsilon (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' + \tfrac{1}{2} \varepsilon^2 (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}'' + O(\varepsilon^3). \tag{6.1}$$

Notice that in this and all following expressions, the multiplication vector times matrix (no $\cdot$) takes precedence over the explicit dot product. Thus,

$$\begin{aligned} \boldsymbol{P} \circ \boldsymbol{\eta}_\varepsilon \cdot \dot{\boldsymbol{\eta}}_\varepsilon = {} & \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \varepsilon (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \dot{\boldsymbol{\eta}} + \varepsilon \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}}' \\ & + \tfrac{1}{2} \varepsilon^2 \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}}'' + \tfrac{1}{2} \varepsilon^2 (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}'' \cdot \dot{\boldsymbol{\eta}} + \varepsilon^2 (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \dot{\boldsymbol{\eta}}' + O(\varepsilon^3). \end{aligned} \tag{6.2}$$

We can extract from this expression some full time derivatives which do not contribute to the variational principle. For any vector $\boldsymbol{w}$,

$$\partial_t (\boldsymbol{P} \circ \boldsymbol{\eta} \cdot \boldsymbol{w}) = (\nabla \boldsymbol{P})^T \circ \boldsymbol{\eta}\, \boldsymbol{w} \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{w}}, \tag{6.3}$$

so that

$$\begin{aligned} \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{w}} + (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{w} \cdot \dot{\boldsymbol{\eta}} &= \left(\nabla \boldsymbol{P} - (\nabla \boldsymbol{P})^T\right) \circ \boldsymbol{\eta}\, \boldsymbol{w} \cdot \dot{\boldsymbol{\eta}} + \partial_t (\boldsymbol{P} \circ \boldsymbol{\eta} \cdot \boldsymbol{w}) \\ &= \boldsymbol{w}^\perp \cdot \dot{\boldsymbol{\eta}} + \partial_t (\boldsymbol{P} \circ \boldsymbol{\eta} \cdot \boldsymbol{w}). \end{aligned} \tag{6.4}$$

Similarly, we compute, again under the assumption that f is constant (when f is arbitrary, the additional terms we obtain do not combine in the same way),

$$\partial_t \left((\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \boldsymbol{\eta}'\right) = (\nabla \boldsymbol{P})^T \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \dot{\boldsymbol{\eta}}' + (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \dot{\boldsymbol{\eta}}', \tag{6.5}$$

so that

$$\begin{aligned} (\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \dot{\boldsymbol{\eta}}' &= \tfrac{1}{2} \left(\nabla \boldsymbol{P} - (\nabla \boldsymbol{P})^T\right) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \dot{\boldsymbol{\eta}}' + \tfrac{1}{2} \partial_t \left((\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \boldsymbol{\eta}'\right) \\ &= \tfrac{1}{2} \boldsymbol{\eta}'^\perp \cdot \dot{\boldsymbol{\eta}}' + \tfrac{1}{2} \partial_t \left((\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \boldsymbol{\eta}'\right). \end{aligned} \tag{6.6}$$

We now apply (6.4) with $\boldsymbol{w} = \boldsymbol{\eta}'$ and $\boldsymbol{w} = \boldsymbol{\eta}''$ respectively, and (6.6) to rewrite (6.2) as

$$\begin{aligned} \boldsymbol{P} \circ \boldsymbol{\eta}_\varepsilon \cdot \dot{\boldsymbol{\eta}}_\varepsilon &= \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \varepsilon \boldsymbol{\eta}'^\perp \cdot \dot{\boldsymbol{\eta}} + \varepsilon \partial_t (\boldsymbol{P} \circ \boldsymbol{\eta} \cdot \boldsymbol{\eta}') + \tfrac{1}{2} \varepsilon^2 \boldsymbol{\eta}''^\perp \cdot \dot{\boldsymbol{\eta}} \\ &\quad + \tfrac{1}{2} \varepsilon^2 \partial_t (\boldsymbol{P} \circ \boldsymbol{\eta} \cdot \boldsymbol{\eta}'') + \tfrac{1}{2} \varepsilon^2 \boldsymbol{\eta}'^\perp \cdot \dot{\boldsymbol{\eta}}' + \tfrac{1}{2} \varepsilon^2 \partial_t \left((\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \boldsymbol{\eta}' \cdot \boldsymbol{\eta}'\right) + O(\varepsilon^3) \\ &= \left[\boldsymbol{P} \cdot \boldsymbol{u} + \varepsilon \boldsymbol{u} \cdot \boldsymbol{v}^\perp + \tfrac{1}{2} \varepsilon^2 \left(\boldsymbol{u} \cdot (\boldsymbol{v}' + \nabla \boldsymbol{v}\, \boldsymbol{v})^\perp + \boldsymbol{v}^\perp \cdot (\dot{\boldsymbol{v}} + \nabla \boldsymbol{v}\, \boldsymbol{u})\right)\right] \circ \boldsymbol{\eta} \\ &\quad + O(\varepsilon^3) + \dot{F}, \end{aligned} \tag{6.7}$$

where $\dot{F}$ is a total time derivative which does not contribute to the variational principle, and will be dropped henceforth.

The potential energy term is easily expanded by noting that (4.6) and (4.7) combine to $\boldsymbol{\eta}'_\varepsilon = \boldsymbol{v}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon$, so that the Liouville theorem for the flow of $\boldsymbol{v}_\varepsilon$ reads

$$J'_\varepsilon = (\nabla \cdot \boldsymbol{v}_\varepsilon) \circ \boldsymbol{\eta}_\varepsilon J_\varepsilon. \tag{6.8}$$

After differentiating with respect to ε, setting $\varepsilon = 0$ yields the relations

$$J' = (\nabla \cdot \boldsymbol{v}) \circ \boldsymbol{\eta}\, J, \tag{6.9}$$

$$J'' = \left[\nabla \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot \nabla\nabla \cdot \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})^2\right] \circ \boldsymbol{\eta}\, J. \tag{6.10}$$

Thus,

$$J_\varepsilon = J \left[1 + \varepsilon \nabla \cdot \boldsymbol{v} + \tfrac{1}{2} \varepsilon^2 \left(\nabla \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot \nabla\nabla \cdot \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})^2\right)\right] \circ \boldsymbol{\eta} + O(\varepsilon^3). \tag{6.11}$$

This power series is easily inverted, and setting $J^{-1} \equiv h \circ \eta$, we find

$$h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \equiv J_\varepsilon^{-1} = \left(h\left[1 - \varepsilon \nabla \cdot \boldsymbol{v} - \tfrac{1}{2}\varepsilon^2 \left(\nabla \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot \nabla\nabla \cdot \boldsymbol{v} - (\nabla \cdot \boldsymbol{v})^2\right)\right]\right) \circ \boldsymbol{\eta} + O(\varepsilon^3). \quad (6.12)$$

7. Canonical coordinates

For the new coordinates to be canonical, we require that the expanded Lagrangian has the form

$$L_\varepsilon = \int \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} \, \mathrm{d}\boldsymbol{a} - H_\varepsilon + O(\varepsilon^3). \quad (7.1)$$

This corresponds to keeping only the upper-left sub-matrix of the symplectic matrix in (3.4), which itself is symplectic.

At order ε we therefore require that

$$\int (\boldsymbol{u} \cdot \boldsymbol{v}^\perp + \boldsymbol{u} \cdot \nabla^\perp h)\, h \, \mathrm{d}\boldsymbol{x} = 0, \quad (7.2)$$

so that it is sufficient to set

$$\boldsymbol{v} = -\nabla h = \boldsymbol{u}^\perp + O(\varepsilon). \quad (7.3)$$

The last equality follows directly from a full expansion of the constraint (5.1). Notice that $\boldsymbol{v} = \boldsymbol{u}^\perp$ only in the limit when $\varepsilon \to 0$.

At the next order, we need

$$\int \left(\tfrac{1}{2}\boldsymbol{u} \cdot (\boldsymbol{v}' + \nabla\boldsymbol{v}\,\boldsymbol{v})^\perp + \tfrac{1}{2}\boldsymbol{v}^\perp \cdot (\dot{\boldsymbol{v}} + \nabla\boldsymbol{v}\,\boldsymbol{u}) + \boldsymbol{u} \cdot (\boldsymbol{w} + \dot{\boldsymbol{v}} + \nabla\boldsymbol{v}\,\boldsymbol{u})\right) h \, \mathrm{d}\boldsymbol{x} = 0. \quad (7.4)$$

We can use (7.3) to express $\boldsymbol{u}$ and $\boldsymbol{v}$ in terms of h within the required order of accuracy. Notice that the terms involving $\dot{\boldsymbol{v}}$ do not contribute, as

$$\int \nabla^\perp h \cdot \nabla \dot{h}\, h \, \mathrm{d}\boldsymbol{x} = -\int \dot{h}\left(h \nabla \cdot \nabla^\perp h + \nabla h \cdot \nabla^\perp h\right) \mathrm{d}\boldsymbol{a} = 0. \quad (7.5)$$

Therefore, a sufficient condition for (7.4) to hold is

$$(\boldsymbol{v}' + \nabla\nabla h\, \nabla h)^\perp - \nabla\nabla h\, \nabla^\perp h + 2\boldsymbol{w} = 0, \quad (7.6)$$

or

$$\boldsymbol{v}' = 2\boldsymbol{w}^\perp - 2\,\nabla\nabla^\perp h\, \nabla^\perp h - \Delta h\, \nabla h. \quad (7.7)$$

We can now compute the expanded Hamiltonian up to order ε^2, namely

$$H_\varepsilon = H_0 + \varepsilon H_1 + \tfrac{1}{2}\varepsilon^2 H_2 + O(\varepsilon^3), \quad (7.8)$$

where

$$H_0 = \tfrac{1}{2}\int h \circ \boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} = \tfrac{1}{2}\int h^2 \, \mathrm{d}\boldsymbol{x}, \quad (7.9)$$

$$H_1 = \tfrac{1}{2}\int \left(|\nabla h|^2 + h\Delta h\right) \circ \boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} = -\tfrac{1}{2}\int h\,|\nabla h|^2 \, \mathrm{d}\boldsymbol{x}, \quad (7.10)$$

and

$$\begin{aligned} H_2 &= \int 2\,(\boldsymbol{w} \cdot \nabla^\perp h) \circ \boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} - \tfrac{1}{2}\int \left[h\left(\nabla \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot \nabla\nabla \cdot \boldsymbol{v} - (\nabla \cdot \boldsymbol{v})^2\right)\right] \circ \boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} \\ &= \int \boldsymbol{w} \cdot \nabla^\perp h^2 \, \mathrm{d}\boldsymbol{x} - \int h^2 \left[\nabla \cdot \boldsymbol{w}^\perp - \nabla\nabla^\perp h : \nabla^\perp\nabla h - (\Delta h)^2\right] \mathrm{d}\boldsymbol{x} \\ &= \int h^2\, |\mathrm{Hess}\, h|^2 \, \mathrm{d}\boldsymbol{x}. \end{aligned} \quad (7.11)$$

Altogether, the fully transformed Lagrangian up to order ε^2 reads

$$L = \int \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} \,\mathrm{d}\boldsymbol{a} - H \tag{7.12a}$$

$$H = \tfrac{1}{2} \int \left[h^2 - \varepsilon\, h\, |\nabla h|^2 + \varepsilon^2\, h^2\, |\mathrm{Hess}\, h|^2\right] \,\mathrm{d}\boldsymbol{x}. \tag{7.12b}$$

8. Equations of motion

Taking arbitrary variations of the flow map, we find

$$\begin{aligned} \delta \int \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} \,\mathrm{d}\boldsymbol{a} &= \int [(\nabla \boldsymbol{P}) \circ \boldsymbol{\eta}\, \delta\boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{P} \circ \boldsymbol{\eta} \cdot \delta\dot{\boldsymbol{\eta}}] \,\mathrm{d}\boldsymbol{a} \\ &= \int \left[((\nabla \boldsymbol{P})^T - \nabla \boldsymbol{P})\, \boldsymbol{u}\right] \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta} \,\mathrm{d}\boldsymbol{a} \\ &= -\int h\, \boldsymbol{u}^\perp \cdot \delta\boldsymbol{\eta} \circ \boldsymbol{\eta}^{-1} \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{8.1}$$

Moreover, for any function $\phi = \phi(\boldsymbol{x})$

$$\int \phi\, \delta h \,\mathrm{d}\boldsymbol{x} = \int h\, \nabla\phi \cdot \delta\boldsymbol{\eta} \circ \boldsymbol{\eta}^{-1} \,\mathrm{d}\boldsymbol{x}. \tag{8.2}$$

This last identity can be derived as follows. First note that

$$\nabla\boldsymbol{\eta}^\perp : \nabla^\perp \boldsymbol{\eta} = -2 \det \nabla\boldsymbol{\eta}, \tag{8.3}$$

so that

$$-2 = h \circ \boldsymbol{\eta}\, \nabla\boldsymbol{\eta}^\perp : \nabla^\perp \boldsymbol{\eta}. \tag{8.4}$$

By taking the variation of this identity and after minor simplification we obtain

$$\delta h \circ \boldsymbol{\eta} = h^2 \circ \boldsymbol{\eta}\, \nabla\boldsymbol{\eta}^\perp : \nabla^\perp \delta\boldsymbol{\eta} - (\nabla h) \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta}. \tag{8.5}$$

We can now insert this expression for δh into the left-hand side of (8.2), and integrate by parts:

$$\begin{aligned} \int \phi\, \delta h \,\mathrm{d}\boldsymbol{x} &= \int \left[(h\phi) \circ \boldsymbol{\eta}\;\; \nabla\boldsymbol{\eta}^\perp : \nabla^\perp \delta\boldsymbol{\eta} - \left(\phi\, \frac{\nabla h}{h}\right) \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta}\right] \,\mathrm{d}\boldsymbol{a} \\ &= -\int \left[(\nabla(h\phi)) \circ \boldsymbol{\eta} \cdot \nabla^\perp \boldsymbol{\eta}\, (\nabla\boldsymbol{\eta}^\perp)^T\, \delta\boldsymbol{\eta} + \left(\phi\, \frac{\nabla h}{h}\right) \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta}\right] \,\mathrm{d}\boldsymbol{a}. \end{aligned} \tag{8.6}$$

Since $\nabla^\perp \boldsymbol{\eta}\, (\nabla\boldsymbol{\eta}^\perp)^T = -\boldsymbol{I} \det \nabla\boldsymbol{\eta} = -\boldsymbol{I}\, h^{-1} \circ \boldsymbol{\eta}$, (8.2) is proved. We use this identity to compute the variation of the transformed Hamiltonian. A direct computation using integration by parts gives

$$\delta H = \int \phi\, \delta h \,\mathrm{d}\boldsymbol{x} = \int h\, \nabla\phi \cdot \delta\boldsymbol{\eta} \circ \boldsymbol{\eta}^{-1} \,\mathrm{d}\boldsymbol{x}, \tag{8.7}$$

where

$$\phi = h + \tfrac{1}{2}\, \varepsilon\, |\nabla h|^2 + \varepsilon\, h\, \Delta h + \varepsilon^2\, h\, |\mathrm{Hess}\, h|^2 + \varepsilon^2\, \mathrm{Hess} : (h^2\, \mathrm{Hess}\, h). \tag{8.8}$$

The variational principle $\delta L = 0$ then gives $\boldsymbol{u} = \nabla^\perp \phi$, so that the conservation-of-mass equation reads

$$\partial_t h + \nabla^\perp \cdot (\phi\, \nabla h) = 0. \tag{8.9}$$

The evolution clearly preserves h along particle paths. This can be viewed as a conservation law for the potential vorticity:

$$q \circ \boldsymbol{\eta} = \frac{1}{h \circ \boldsymbol{\eta}} = \frac{1 + \nabla^{\perp} \boldsymbol{u}_{\text{geostrophic}}}{h_{\varepsilon}} \circ \boldsymbol{\eta}_{\varepsilon} + O(\varepsilon^2), \tag{8.10}$$

as is the case for Salmon's (1985) model. Note that this conservation law does not depend on the exact definition of ϕ, but only on the symplectic structure of the resulting Hamilton system which has not been altered by going to second order in the transformation. In fact, as has been noted by many authors, potential vorticity conservation can be independently derived by using the particle relabelling symmetry of the Hamiltonian (see, for example, Salmon 1996 or Bridges, Hydon & Reich 2001).

9. Discussion

By maintaining $O(\varepsilon^2)$ accuracy in the transformation to canonical coordinates we have derived a new model for large-scale geostrophic flow that has a positive-definite energy with respect to the coordinate system in which we compute. This is generally important, because as soon as the different components of the energy have opposite signs, none of them is *a priori* bounded even though the full Hamiltonian is conserved. For example, numerical experiments performed by Shepherd & Ford (2001) on a model for a single thermally active layer in a stratified ocean which is closely related to the simpler shallow-water equations have shown that without such *a priori* control rapid formation of small-scale structures occurs and is difficult to control. We must note, however, that the particular situation we discuss in this paper is special because even if we truncated the Hamiltonian at $O(\varepsilon)$, the point-wise conservation of h would imply conservation of H_0, thereby constraining H_1 as well. For this reason we do not expect this particular new model to be drastically more stable than Salmon's (1985) model. Nonetheless, we believe that this work is useful for several reasons.

1. The principle of recovering definiteness by including higher-order terms in an asymptotic expansion is much more general. In particular, it can be applied to reduced models of the primitive equations of ocean dynamics, which would cover the situation discussed in Salmon (1996) and Shepherd & Ford (2001).

2. The introduction of sign-definite higher-order terms into the Hamiltonian is reminiscent of Hamiltonians corresponding to Lagrangian averaged Euler equations (also known as Euler-α equations or equations of inviscid second-grade fluids), which have been extensively discussed in recent years (Holm, Marsden & Ratiu 1998; Holm 1999; Marsden & Shkoller 2000, 2001; Oliver & Shkoller 2001). The ideas developed there may eventually lead to a more direct physical, as opposed to structural, interpretation of the effect of our approximation.

3. Taking the transformation to order-ε^2 accuracy may be philosophically correct: if we accept the picture that the large-scale semi-geostrophic approximation is a projection onto a slow phase-space manifold which is only accurate to first order, we should be *more* accurate when seeking canonical coordinates on this slow manifold, lest we risk introducing fast dynamics back into the system at the very order they initially were removed.

4. Asymptotic models in the small Rossby number are, as far as scaling arguments go, only valid for times of order one – much shorter than time scales of physical interest. On the other hand, semi-geostrophic models in practice often perform adequately over much longer intervals of time. While this general phenomenon is not

fully understood, we propose a link to structure preservation in the geometric as well as an analytic sense: Not only is the reduced model Hamiltonian, it also inherits a modified definite energy law which should prove crucial, in the general case, for rendering such models well posed.

5. The canonical coordinate theorem of McIntyre & Roulstone (1996, 2001) gives a general necessary condition for relating the constraint velocity to the change of variables on the slow manifold. Although their explicit example is – as in Salmon – only a first-order ansatz for the change of variables, it will be interesting to see what class of models one can obtain by performing our approach in their generality.

The authors thank the referees for the careful reading of the manuscript and insightful comments. Part of this work was completed while M.O. was visiting Imperial College and Heriot–Watt University on a London Mathematical Society Scheme 2 visitors grant.

REFERENCES

Bridges, T., Hydon, P. & Reich, S. 2001 Vorticity and symplecticity in Lagrangian fluid dynamics. Preprint.

Ford, R., Malham, S. J. A. & Oliver, M. 2001 The averaged large-scale semi-geostrophic equations. In preparation.

Holm, D. D. 1999 Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion. *Physica* D **133**, 215–269.

Holm, D. D., Marsden, J. E. & Ratiu, T. S. 1998 Euler-Poincaré equations and semidirect products with applications to continuum theories. *Adv. Maths* **137**, 1–81.

Marsden, J. E. & Shkoller, S. 2000 The anisotropic averaged Euler equations. Preprint, `arXiv:math.AP/0005033`.

Marsden, J. E. & Shkoller, S. 2001 Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains. *Phil. Trans. R. Soc. Lond*. A **359**, 1449–1468.

McIntyre, M. E. & Roulstone, I. 1996 Hamiltonian balanced models: constraints, slow manifolds and velocity splitting. Preprint.

McIntyre, M. E. & Roulstone, I. 2001 Are there higher-accuracy analogues of semigeostrophic theory? In *Large-scale Atmosphere–Ocean Dynamics: II: Geometric Methods and Models* (ed. I. Roulstone & J. Norbury). Cambridge University Press (to appear).

Oliver, M. & Shkoller, S. 2001 The vortex blob method as a second-grade non-Newtonian fluid. *Commun. Partial Diffl Equat*. **26**, 295–314.

Salmon, R. 1985 New equations for nearly geostrophic flow. *J. Fluid Mech.* **153**, 461–477.

Salmon, R. 1988 Semi-geostrophic theory as a Dirac-bracket projection. *J. Fluid Mech.* **196**, 345–358.

Salmon, R. 1996 Large-scale semi-geostrophic equations for use in ocean circulation models. *J. Fluid Mech.* **318**, 85–105.

Shepherd, J. R. & Ford, R. 2001 Dynamics of an oceanic model with a single thermally-active layer. *J. Fluid Mech.* (submitted).